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ALEXANDER POLYNOMIALS OF SIMPLE-RIBBON KNOTS

Kengo KISHIMOTO, Tetsuo SHIBUYA, Tatsuya TSUKAMOTO and Tsuneo ISHIKAWA

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Abstract

In [4], we introduced special types of fusions, so called simple-ribbon fusions on links. A knot obtained from the trivial knot by a finite sequence of simple-ribbon fusions is called a simple-ribbon knot. Every ribbon knot with ≤ 9 crossings is a simple-ribbon knot. In this paper, we give a formula for the Alexander polynomials of simple-ribbon knots. Using the formula, we determine if a knot with 10 crossings is a simple-ribbon knot. Every simple-ribbon fusion can be realized by "elementary" simple-ribbon fusions. We call a knot an m-simple-ribbon knot if the knot is obtained from the trivial knot by a finite sequence of elementary m-simple-ribbon fusions for a fixed positive integer m. We provide a condition for a simple-ribbon knot to be both of an m-simple-ribbon knot and an n-simple-ribbon knot for positive integers m and n.

1. Introduction

Knots and links are assumed to be ordered and oriented, and they are considered up to ambient isotopy in an oriented 3-sphere S^3. In [4], we introduced special types of fusions, so called simple-ribbon fusions. A (m-)*ribbon fusion* on a link ℓ is an *m-fusion* ([3, Definition 13.1.1]) on the split union of ℓ and an m-component trivial link $\mathcal{O}$ such that each component of $\mathcal{O}$ is attached to a component of ℓ by a single band. Note that any knot obtained from the trivial knot by a finite sequence of ribbon fusions is a *ribbon knot* ([3, Definition 13.1.9]), and that any ribbon knot can be obtained from the trivial knot by ribbon fusions. Here we only define an elementary simple-ribbon fusion. A general simple-ribbon fusion can be realized by elementary simple-ribbon fusions. Refer [4] for precise definition.

Let ℓ be a link and $\mathcal{O} = O_1 \cup \cdots \cup O_m$ the m-component trivial link which is split from ℓ. Let $\mathcal{D} = D_1 \cup \cdots \cup D_m$ be a disjoint union of non-singular disks with $\partial D_i = O_i$ and $D_i \cap \ell = \emptyset$ ($i = 1, \cdots, m$), and let $\mathcal{B} = B_1 \cup \cdots \cup B_m$ be a disjoint union of disks for an m-fusion, called *bands*, on the split union of ℓ and $\mathcal{O}$ satisfying the following (see Figure 1 for example):

(i) $B_i \cap \ell = \partial B_i \cap \ell = \{$ a single arc $\}$;

(ii) $B_i \cap \mathcal{O} = \partial B_i \cap O_i = \{$ a single arc $\}$; and

(iii) $B_i \cap \operatorname{int} \mathcal{D} = B_i \cap \operatorname{int} D_{i+1} = \{$ a single arc of ribbon type $\}$.

Let L be a link obtained from the split union of ℓ and $\mathcal{O}$ by the m-fusion along $\mathcal{B}$, i.e., $L = (\ell \cup \mathcal{O} \cup \partial\mathcal{B}) - \operatorname{int}(\mathcal{B} \cap \ell) - \operatorname{int}(\mathcal{B} \cap \mathcal{O})$. Then we say that L is obtained from ℓ by an *elementary* (m-)*simple-ribbon fusion* or an *elementary* (m-)SR-fusion (*with respect to*

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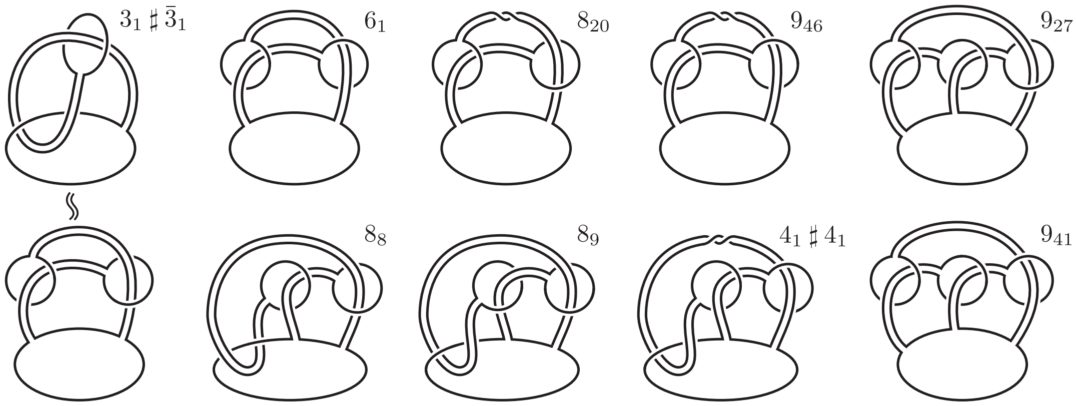

Fig. 1. ribbon knots with less than or equal to nine crossings

$\mathcal{D} \cup \mathcal{B}$). If a knot K is obtained from the trivial knot O by a finite sequence of elementary SR-fusions, then we call K a *simple-ribbon knot* (or an SR-knot). Since an elementary SR-fusion is a ribbon fusion, any SR-knot is a ribbon knot. We also call the trivial knot an SR-knot. As illustrated in Figure 1, all the ribbon knots with ≤ 9 crossings are SR-knots.

Let $\dot{D}_i$ and $\dot{B}_i$ be disks and $f : \cup_i \left(\dot{D}_i \cup \dot{B}_i\right) \to S^3$ an immersion such that $f(\dot{D}_i) = D_i$ and $f(\dot{B}_i) = B_i$. We denote the arc of int $D_i \cap B_{i-1}$ by α_i and let $B_{i,1}$ and $B_{i,2}$ be the subdisks of B_i such that $B_{i,1} \cup B_{i,2} = B_i$, $B_{i,1} \cap B_{i,2} = \alpha_{i+1}$, and $B_{i,1} \cap \partial D_i \neq \emptyset$. Take a point b_i on int α_i ($i = 1, \ldots, m$) and an arc β_i on $D_i \cup B_{i,1}$ so that $\beta_i \cap (\alpha_i \cup \alpha_{i+1}) = \partial\beta_i = b_i \cup b_{i+1}$ and oriented from b_{i+1} to b_i (see Figure 2). Then $\beta = \cup_i \beta_i$ is an oriented simple loop and we call β an *attendant knot* of $\mathcal{D} \cup \mathcal{B}$. Moreover, we denote the pre-images of α_i (resp. b_i) on $\dot{D}_i$ and $\dot{B}_{i-1}$ by $\dot{\alpha}_i$ and $\ddot{\alpha}_i$ (resp. $\dot{b}_i$ and $\ddot{b}_i$), respectively.

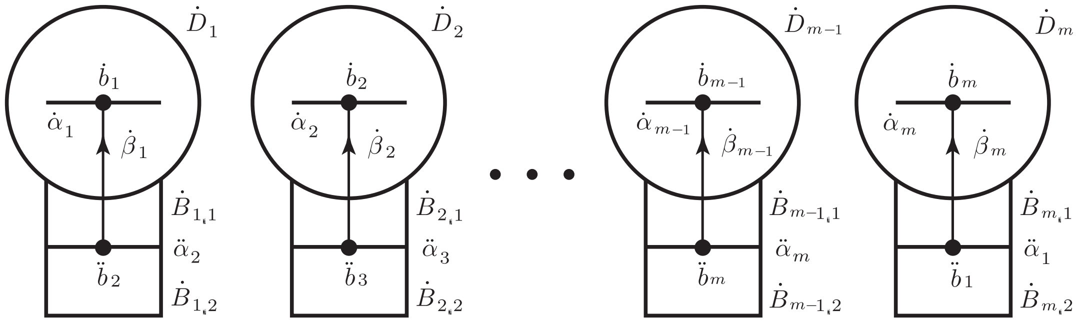

Fig. 2

$\mathcal{D} \cup \mathcal{B}$ is oriented so that induced orientations on boundaries are compatible with the orientation of ℓ. Then we can see that each band B_i intersects with D_{i+1} in two ways, i.e. when we pass through α_{i+1} from $B_{i,2}$ to $B_{i,1}$, we pass through D_{i+1} either from the negative side to the positive side of D_{i+1}, or from the positive side to the negative side of D_{i+1}. In the former and latter cases, we say that B_i is *positive* and *negative*, respectively. Then we have the following.

Theorem 1.1. *Let K be a knot obtained from a knot k by an elementary m-SR-fusion with an attendant knot β and with p positive bands. Let $l = \mathrm{lk}(\beta, k)$ and $\varphi(t\,; m, p, l) = (1-t)^m - t^l(-t)^p$. Then we have the following.*

$$\Delta_K(t) \doteq \Delta_k(t)\,\varphi(t\,; m, p, l)\,\varphi(t^{-1}\,; m, p, l) \tag{1.1}$$

REMARK. We can also write $\Delta_K(t)$ as $\Delta_k(t)\,\varphi(t\,; m, p, l)\,\varphi(t\,; m, m-p, -l)$, i.e.

$$\Delta_K(t) \doteq \Delta_k(t)\,\{(1-t)^m - t^l(-t)^p\}\,\{(1-t)^m - t^{-l}(-t)^{m-p}\} \tag{1.2}$$

Corollary 1.2. *Let K be a knot obtained from a knot k by a finite sequence of elementary SR-fusions, i.e., there exists a finite sequence $k = K_0, K_1, \ldots, K_N = K$ of knots such that K_i is obtained from K_{i-1} by an elementary m_i-SR-fusion with an attendant knot β_i and with p_i positive bands ($i = 1, \ldots, N$). Let $l_i = \mathrm{lk}(\beta_i, K_{i-1})$ and $\varphi(t\,; m_i, p_i, l_i) = (1-t)^{m_i} - t^{l_i}(-t)^{p_i}$. Then we have the following.*

$$\Delta_K(t) \doteq \Delta_k(t) \prod_{i=1}^{N} \varphi(t\,; m_i, p_i, l_i)\,\varphi(t^{-1}\,; m_i, p_i, l_i) \tag{1.3}$$

As mentioned in the beginning, all the ribbon knots with ≤ 9 crossings are SR-knots. Using Corollary 1.2, we can determine if a ribbon knot with 10 crossings is an SR-knot. To do this, we use a value derived from the Alexander polynomial. For a knot K, let $\Delta'_K(t)$ be the polynomial such that $\Delta'_K(t) \doteq \Delta_K(t)$ and $\Delta'_K(0) \neq 0$. Then define $\delta_2(K)$ as 0 if $|\Delta'_K(2)| = 0$ and as the largest odd factor of $|\Delta'_K(2)|$ if $|\Delta'_K(2)| \neq 0$. Note that if K is a simple-ribbon knot, then $\delta_2(K)$ is a product of the integers of the form $2^s \pm 1$ ($s = 0, 1, 2, \ldots$) from Corollary 1.2.

Lemma 1.3. *If K is a simple-ribbon knot such that $\delta_2(K) = 1$, then we have the following for a non-negative integer n.*

$$\Delta'_K(t) = 1 \text{ or } (1 - 6t + 11t^2 - 6t^3 + t^4)^n \tag{1.4}$$

Proof. Since K is a simple-ribbon knot, we have the following from Corollary 1.2, where $N\ (\geq 1)$, $m_i\ (\geq 1)$, $p_i\ (0 \leq p_i \leq m_i)$, and l_i are integers ($i = 1, 2, \ldots, N$).

$$\begin{aligned}\Delta_K(t) &\doteq \textstyle\prod_{i=1}^{N}\{(1-t)^{m_i} - t^{l_i}(-t)^{p_i}\}\{(1-t)^{m_i} - t^{-l_i}(-t)^{m_i-p_i}\}\\ &\doteq \textstyle\prod_{i=1}^{N}\{t^{p_i+l_i} + (-1)^{m_i-(p_i+1)}(t-1)^{m_i}\}\{t^{m_i-(p_i+l_i)} + (-1)^{p_i+1}(t-1)^{m_i}\}\end{aligned}$$

Let $g_i(t) = t^{p_i+l_i} + (-1)^{m_i-(p_i+1)}(t-1)^{m_i}$ and $h_i(t) = t^{m_i-(p_i+l_i)} + (-1)^{p_i+1}(t-1)^{m_i}$. Then we have that $\Delta'_K(2) = 2^q \prod_{i=1}^{N} g_i(2)h_i(2)$ for an integer q. Since $\delta_2(K) = 1$, each of $|g_i(2)|$ and $|h_i(2)|$ is a power of 2, and thus $2^{-1} = |2^{-1} - 1|$, $2 = 2^0 + 1$, or $1 = 2^1 - 1$ ($i = 1, 2, \ldots, N$). Thus, each of $p_i + l_i$ and $m_i - (p_i + l_i)$ is -1, 0, or 1 for each i, and hence $m_i = (p_i + l_i) + (m_i - (p_i + l_i))$ is 1 or 2, since $m_i > 0$. Therefore we have that $(g_i(2), h_i(2), m_i) = (2^0 + 1, 2^1 - 1, 1)$, $(2^1 - 1, 2^0 + 1, 1)$, or $(2^1 - 1, 2^1 - 1, 2)$. In the first two cases and the last case, we have that $g_i(t)h_i(t) = \{t^0 + (t-1)\}\{t^1 - (t-1)\} = t$ and $g_i(t)h_i(t) = \{t - (t-1)^2\}^2 = 1 - 6t + 11t^2 - 6t^3 + t^4$, respectively. Hence we obtain the conclusion. □

Proposition 1.4. *Among the* 16 *ribbon knots with* 10 *crossings,* 10_{42}, 10_{75}, 10_{87}, 10_{99}, 10_{129}, 10_{137}, 10_{140}, 10_{153}, *and* 10_{155} *are simple-ribbon knots and* 10_3, 10_{22}, 10_{35}, 10_{48}, 10_{123}, $5_1 \sharp 5_1^*$, *and* $5_2 \sharp 5_2^*$ *are not.*

Proof. The former statement is from Figure 3. To show the latter statement, we consider δ_2 for each knot. Since $\delta_2(10_{22}) = 11$, $\delta_2(10_{48}) = 7\times 13 = 1\times 91$, and $\delta_2(5_1 \sharp 5_1^*) = 11\times 11 = 1\times 121$ from Table 1 and none of 11, 13, 91, and 121 is $2^s \pm 1$ for a non-negative integer s, we know that these 3 knots are not simple-ribbon knots. For the other 4 knots, we have that $\delta_2(10_3) = \delta_2(10_{35}) = \delta_2(10_{123}) = \delta_2(5_2 \sharp 5_2^*) = 1$, and the following from Table 1. Hence we know that they are not simple-ribbon knots from Lemma 1.3.

$$\Delta'_{10_3}(t) = 6 - 13t + 6t^2, \qquad \Delta'_{10_{35}}(t) = 2 - 12t + 21t^2 - 12t^3 + 2t^4,$$

$$\Delta'_{10_{123}}(t) = (1 - 3t + 3t^2 - 3t^3 + t^4)^2, \qquad \Delta'_{5_2 \sharp 5_2^*}(t) = 4 - 12t + 17t^2 - 12t^3 + 4t^4 \qquad \square$$

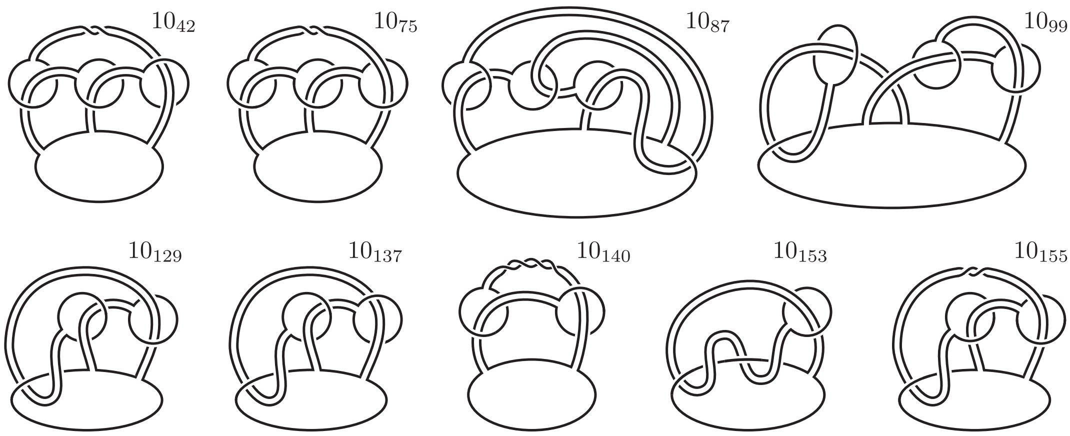

Fig. 3

Note that the above proof of Proposition 1.4 implies that for any ribbon knot K with ≤ 10 crossings, if $\Delta_K(t)$ can be written as equation (1.3), then K is a simple-ribbon knot. However, it does not hold in general.

Theorem 1.5. *For any polynomial* $\Delta(t) = \prod_{i=1}^{N} \varphi(t\,; m_i, p_i, l_i)\,\varphi(t^{-1}\,; m_i, p_i, l_i)$*, there exists a ribbon knot whose Alexander polynomial is* $\Delta(t)$ *and which is not a simple-ribbon knot.*

If an SR-knot is obtained from the trivial knot by a finite sequence of elementary m-SR-fusions for a fixed positive integer m, then we call the SR-knot m-SR-knot. For example, 8_9 is a 2-SR-knot and $3_1 \sharp 3_1^*$ is a 1-SR-knot and also a 2-SR-knot as we can see in Figure 1. It is natural to ask if there exists a simple-ribbon knot which is an m-SR-knot and also an n-SR-knot for distinct positive integers m and n other than $3_1 \sharp 3_1^*$. We give a partial answer to this question using equation (1.3). Let m be a positive integer and $\mathcal{K}_m$ the set of non-trivial m-SR-knots. Then we have the following.

Theorem 1.6. *If* $\mathcal{K}_m \cap \mathcal{K}_n \neq \emptyset$ *for positive integers* m *and* n *with* $m > n$*, then we have either that* $(m, n) = (3, 1)$*,* $(3, 2)$*, or* $(2n, n)$*.*

2. Proofs of Theorem 1.1 and Theorem 1.5

Let K be a knot obtained from a knot k by an elementary m-SR-fusion with respect to $\mathcal{D} \cup \mathcal{B}$ with its attendant knot β. Let F be a Seifert surface for k. Here we may take F so that $F \cap \mathcal{D} = \emptyset$. Let $C = F \cup (\mathcal{D} \cup \mathcal{B})$. We first transform C into "standard" position and construct a Seifert surface F_K for K from C in standard position. Then, we calculate $\Delta_K(t)$ using F_K.

We may take F so that the intersections with $\mathcal{D} \cup \mathcal{B}$ are only arcs of int$F$ and $\mathcal{B}$. Then we divide the set of singularities of int$F \cap B_i$ into two: one which consists of int$F \cap B_{i,1}$, denoted by $\mathcal{S}_i$, and the other which consists of int$F \cap B_{i,2}$, denoted by $\mathcal{T}_i$. Thus the set of singularities of C is $\cup_i \alpha_i \cup \cup_i (\mathcal{S}_i \cup \mathcal{T}_i)$. We say that C is *in standard position* if $\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_{m-1} = \emptyset$ and $\mathcal{T}_1 \cup \cdots \cup \mathcal{T}_m = \emptyset$ (see Figure 9 for example). To transform C into standard position, we need the following three transformations. Here note that each transformation changes neither m, p, nor the knot type of β.

Sliding a disk along a band : Deforming D_{i+1} by deformation retraction into a regular neighborhood of B_i and slide D_{i+1} along B_i toward D_i. Here B_{i+1} follows D_{i+1} (see Figure 4 for example). We allow $D_{i+1} \cup B_{i+1}$ to pass through F. Then an additional intersection of B_{i+1} and F is created.

Winding a band along k : Winding B_i along $k = \partial F$ in a regular neighborhood of $B_i \cap k$ either from negative side to positive side or from positive side to negative side (see Figure 5 for example). Here an additional intersection of B_i and F is created.

Tubing F : Removing two disks δ_1 and δ_2 from int F and attatch an annulus $S^1 \times [1, 2]$ so that $S^1 \times \{i\} = \partial \delta_i$ $(i = 1, 2)$ and the result is orientable (see Figure 6 for example).

Proposition 2.1. *Let K be a knot obtained from a knot k by an elementary m-SR-fusion with respect to $\mathcal{D} \cup \mathcal{B}$ with its attendant knot β. Let F be a Seifert surface for k such that $F \cap \mathcal{D} = \emptyset$ and let $C = F \cup (\mathcal{D} \cup \mathcal{B})$. Then we may transform C into standard position by sliding a disk along a band, winding a band along k, and tubing F.*

Proof. First if $\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_{m-1} \neq \emptyset$, then take the smallest index i such that $\mathcal{S}_i \neq \emptyset$ and slide D_{i+1} along B_i just next to D_i so that $\mathcal{S}_i = \emptyset$ (See Figure 4 for example). Then slide D_{j+1} along B_j inductively just next to D_j so that $\mathcal{S}_j = \emptyset$ $(j = i+1, \ldots, m-1)$.

Next if $\mathcal{T}_1 \cup \cdots \cup \mathcal{T}_m \neq \emptyset$, then take an arbitrary $\mathcal{T}_i \neq \emptyset$ and let $t_1, \ldots, t_p$ be its singularities which are placed close to $B_i \cap k$ on B_i in this order. Assume that B_i is oriented as from $B_i \cap k$ towards $B_i \cap D_i$ and let $\sigma(t_j)$ be the signed intersection number of B_i and F at t_j. First wind B_i along k depending on $\sigma(t_j)$ $(j = 1, \ldots, p)$. If $\sigma(t_j) = 1$ (resp. -1), then wind B_i along $k = \partial F$ from negative side to positive side (resp. from positive side to negative side) as illustrated in Figure 5. Here we make these transformations from $j = 1$ to $j = p$ in this order, and notice that each transformation creates a new intersection t'_j with $\sigma(t'_j) = -\sigma(t_j)$. Then make a tubing F so to erase t_j and t'_j from $j = 1$ to $j = p$ in this order as illustrated in Figure 6, and now C is in standard position. □

Proof of Theorem 1.1. Let F be a Seifert surface for k such that $F \cap \mathcal{D} = \emptyset$ and let $C = F \cup (\mathcal{D} \cup \mathcal{B})$. Here we may assume that C is in standard position from Proposition 2.1.

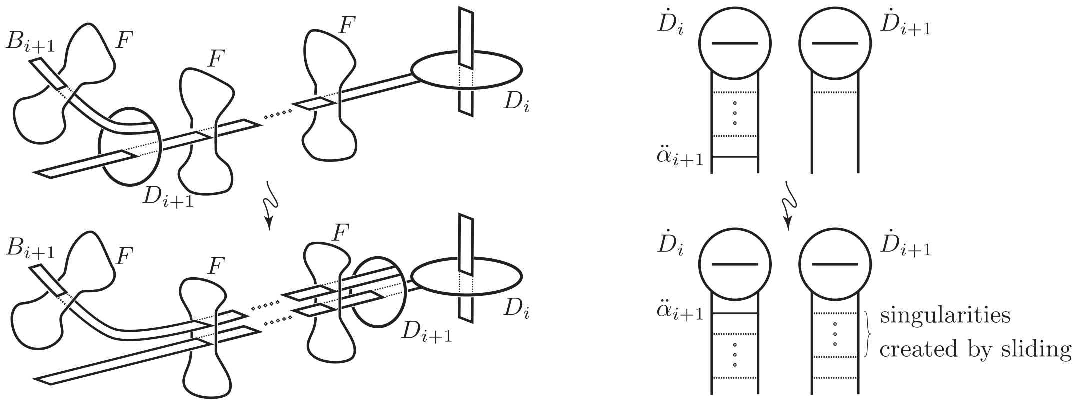

Fig.4

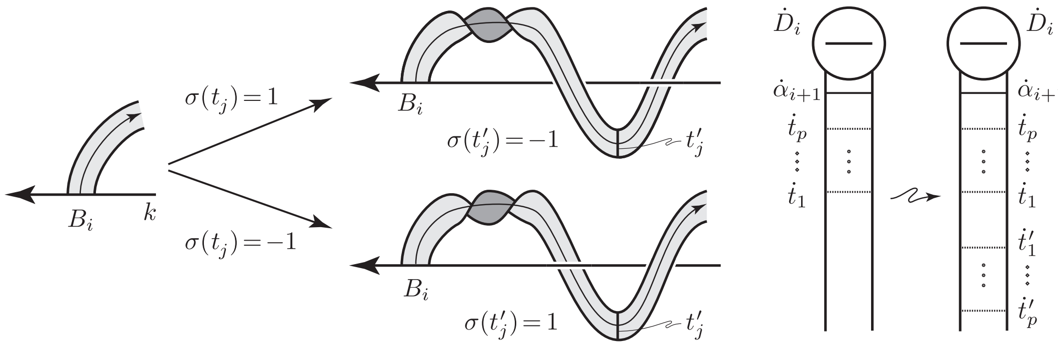

Fig.5

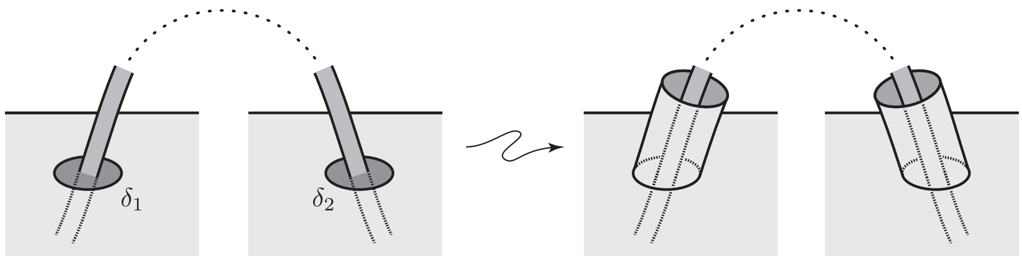

Fig.6

Thus the set of singularities of $\mathcal{C}$ is $\cup_i \alpha_i \cup \mathcal{S}_m$. Erase $\cup_i \alpha_i$ and $\mathcal{S}_m$ to have a Seifert surface F_K for K by orientation preserving cut and deformation as illustrated in the second left of Figure 7 and Figure 8, respectively (see Figure 10 for example of F_K).

Take a basis $x_1, \ldots, x_m, y_1, \ldots, y_{|l|}, z_1, \ldots, z_m, w_1, \ldots, w_{|l|}, u_1, \cdots, u_g$ of $H_1(F_K; \mathbb{Z})$ as illustrated in Figure 7 and Figure 8 (see Figure 10 for example), where $u_1, \cdots, u_g$ is a basis of $H_1(F; \mathbb{Z})$. Then we have the following Seifert matrix M with respect to the basis.

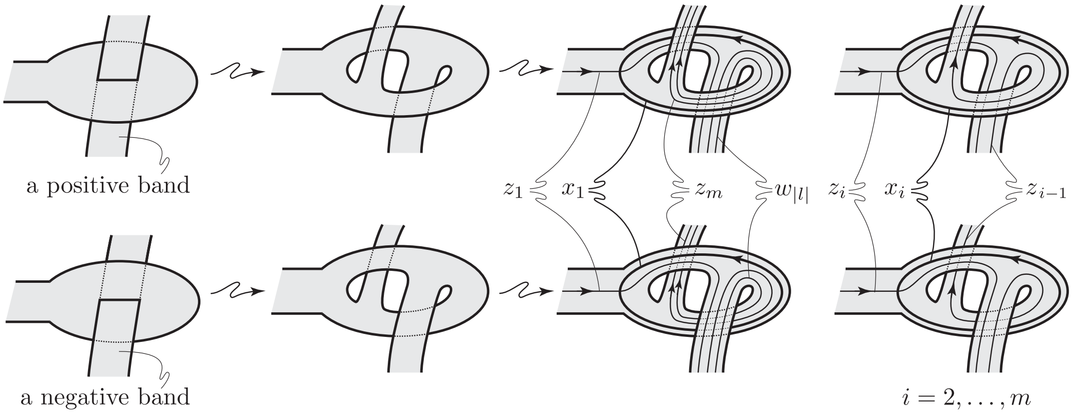

Fig. 7

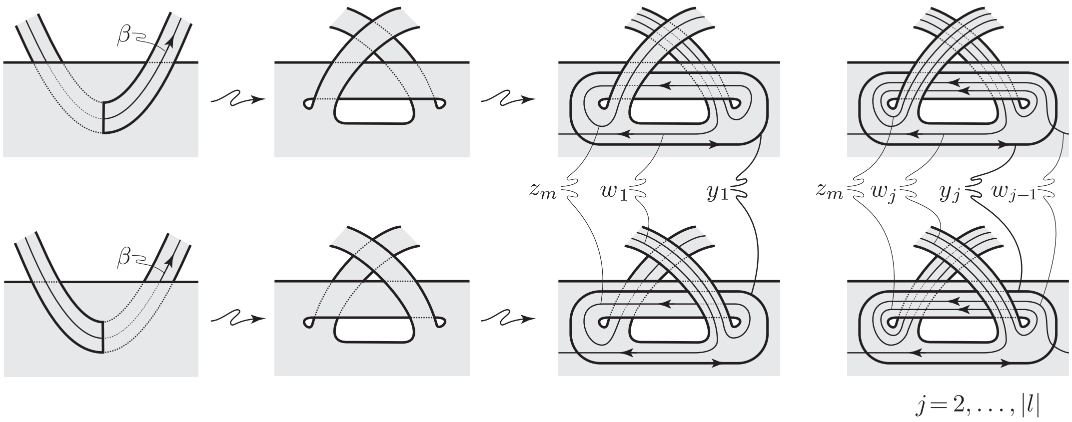

Fig. 8

$$M = \left(\begin{array}{c|c|c} O_{(m+|l|)\times(m+|l|)} & P_{(m+|l|)\times(m+|l|)} & O_{(m+|l|)\times g} \\ \hline Q_{(m+|l|)\times(m+|l|)} & * & * \\ \hline O_{g\times(m+|l|)} & * & M' \end{array}\right) = \left(\begin{array}{c|c|c} O_{(m+|l|)\times(m+|l|)} & \begin{array}{c|c} P^1_{m\times m} & P^2_{m\times|l|} \\ \hline P^3_{|l|\times m} & P^4_{|l|\times|l|} \end{array} & O_{(m+|l|)\times g} \\ \hline \begin{array}{c|c} Q^1_{m\times m} & Q^2_{m\times|l|} \\ \hline Q^3_{|l|\times m} & Q^4_{|l|\times|l|} \end{array} & * & * \\ \hline O_{g\times(m+|l|)} & * & M' \end{array}\right),$$

where M' is a Seifert matrix for k, $O_{s\times t}$ is the $s \times t$ zero matrix,

$$P^1_{m\times m} = (p^1_{ij}) \text{ is an } m \times m \text{ matrix with } p^1_{ij} = \mathrm{lk}(x_i, z^+_j) = \begin{cases} -(\varepsilon_i + 1)/2 & \text{if } i = j \\ \varepsilon_i & \text{if } i = 1 \text{ and } j = m, \\ & \text{or } 2 \le i \le m \text{ and } j = i - 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$P^2_{m\times|l|} = (p^2_{ij}) \text{ is an } m \times |l| \text{ matrix with } p^2_{ij} = \mathrm{lk}(x_i, w^+_j) = \begin{cases} \varepsilon_1 & \text{if } i = 1 \text{ and } j = |l| \\ 0 & \text{otherwise,} \end{cases}$$

$P^3_{|l|\times m} = (p^3_{ij})$ is an $|l| \times m$ matrix with $p^3_{ij} = \mathrm{lk}(y_i, z_j^+) = \begin{cases} \varepsilon & \text{if } j = m \\ 0 & \text{otherwise,} \end{cases}$

$P^4_{|l|\times|l|} = (p^4_{ij})$ is an $|l| \times |l|$ matrix with $p^4_{ij} = \mathrm{lk}(y_i, w_j^+) = \begin{cases} (\varepsilon+1)/2 & \text{if } i = j \\ (\varepsilon-1)/2 & \text{if } 2 \leq i \leq |l| \text{ and } j = i-1 \\ 0 & \text{otherwise,} \end{cases}$

if $l \neq 0$, and $P_{m\times m} = P^1_{m\times m}$ if $l = 0$,

$Q^1_{m\times m} = (q^1_{ij})$ is an $m\times m$ matrix with $q^1_{ij} = \mathrm{lk}(z_i, x_j^+) = \begin{cases} -(\varepsilon_i - 1)/2 & \text{if } i = j \\ \varepsilon_i & \text{if } i = m \text{ and } j = 1, \\ & \text{or } 1 \leq i \leq m-1 \text{ and } j = i+1 \\ 0 & \text{otherwise,} \end{cases}$

$Q^2_{m\times|l|} = (q^2_{ij})$ is an $m \times |l|$ matrix with $q^2_{ij} = \mathrm{lk}(z_i, y_j^+) = \begin{cases} \varepsilon & \text{if } i = m \\ 0 & \text{otherwise,} \end{cases}$

$Q^3_{|l|\times m} = (q^3_{ij})$ is an $|l| \times m$ matrix with $q^3_{ij} = \mathrm{lk}(w_i, x_j^+) = \begin{cases} \varepsilon_1 & \text{if } i = |l| \text{ and } j = 1 \\ 0 & \text{otherwise, and} \end{cases}$

$Q^4_{|l|\times|l|} = (q^4_{ij})$ is an $|l|\times|l|$ matrix with $q^4_{ij} = \mathrm{lk}(w_i, y_j^+) = \begin{cases} (\varepsilon-1)/2 & \text{if } i = j \\ (\varepsilon+1)/2 & \text{if } 1 \leq i \leq |l|-1 \text{ and } j = i+1 \\ 0 & \text{otherwise,} \end{cases}$

if $l \neq 0$, and $Q_{m\times m} = Q^1_{m\times m}$ if $l = 0$, and $\varepsilon = \begin{cases} 1 & \text{if } l \text{ is positive} \\ -1 & \text{if } l \text{ is negative} \end{cases}$, $\varepsilon_i = \begin{cases} 1 & \text{if } B_i \text{ is positive} \\ -1 & \text{if } B_i \text{ is negative} \end{cases}$ $(i = 1, \ldots, m)$. Letting $a = \dfrac{\varepsilon+1}{2}$, $b = \dfrac{\varepsilon-1}{2}$, $a_i = \dfrac{\varepsilon_i+1}{2}$, and $b_i = \dfrac{\varepsilon_i-1}{2}$, we have the following.

$$P = \begin{array}{cc} & \begin{array}{cccccccccc} z_1^+ & z_2^+ & \cdots & z_{m-1}^+ & z_m & w_1^+ & w_2^+ & \cdots & w_{|l|-1}^+ & w_{|l|}^+ \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_{m-1} \\ x_m \\ y_1 \\ y_2 \\ \vdots \\ y_{|l|-1} \\ y_{|l|} \end{array} & \left(\begin{array}{cccccccccc} -a_1 & & & & \varepsilon_1 & & & & & \varepsilon_1 \\ \varepsilon_2 & -a_2 & & & & & & & & \\ & \ddots & \ddots & & & & & & & \\ & & \ddots & -a_{m-1} & & & & & & \\ & & & \varepsilon_m & -a_m & & & & & \\ & & & & \varepsilon & a & & & & \\ & & & & \varepsilon & b & a & & & \\ & & & & \varepsilon & & \ddots & \ddots & & \\ & & & & \varepsilon & & & \ddots & a & \\ & & & & \varepsilon & & & & b & a \end{array}\right) \end{array},$$

$$Q = \begin{array}{cc} & \begin{array}{cccccccccc} x_1^+ & x_2^+ & \cdots & x_{m-1}^+ & x_m & y_1^+ & y_2^+ & \cdots & y_{|l|-1}^+ & y_{|l|}^+ \end{array} \\ \begin{array}{c} z_1 \\ z_2 \\ \vdots \\ z_{m-1} \\ z_m \\ w_1 \\ w_2 \\ \vdots \\ w_{|l|-1} \\ w_{|l|} \end{array} & \left(\begin{array}{cccccccccc} -b_1 & \varepsilon_2 & & & & & & & & \\ & -b_2 & \ddots & & & & & & & \\ & & \ddots & \ddots & & & & & & \\ & & & -b_{m-1} & \varepsilon_m & & & & & \\ \varepsilon_1 & & & & -b_m & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ & & & & & b & a & & & \\ & & & & & & b & a & & \\ & & & & & & & \ddots & \ddots & \\ & & & & & & & & b & a \\ \varepsilon_1 & & & & & & & & & b \end{array}\right) \end{array}$$

Then the Alexander polynomial $\Delta_K(t)$ of K is the product of the Alexander polynomial $\Delta_k(t)$ of k, $|P - t\,Q^T|$, and $|Q - t\,P^T|$.

CLAIM 2.2. We have the following, where $c = a - tb$, $d = b - ta$, $e = \varepsilon(1-t)$, $c_i = a_i - tb_i$, $d_i = b_i - ta_i$, and $e_i = \varepsilon_i(1-t)$.

$$|\,Q{-}tP^T| = d^{|l|}\prod_{i=1}^{m}(-d_i)+(-1)^{|l|+m+1}c^{|l|}\prod_{i=1}^{m}e_i\,,\quad |P{-}t\,Q^T| = c^{|l|}\prod_{i=1}^{m}(-c_i)+(-1)^{|l|+m+1}d^{|l|}\prod_{i=1}^{m}e_i.$$

Proof. First we calculate $|\,P-tQ^T|$ noticing that $e = c + d$. If $l = 0$, then we have that

$$|P-t\,Q^T| = \left|\begin{array}{ccccc} -c_1 & & & & e_1 \\ e_2 & -c_2 & & & \\ & \ddots & \ddots & & \\ & & \ddots & -c_{m-1} & \\ & & & e_m & -c_m \end{array}\right| = c^0\prod_{i=1}^{m}(-c_i) + (-1)^{0+m+1}d^0\prod_{i=1}^{m}e_i.$$

If $|l| = 1$, then we have that

$$|P-\,tQ^T| = \left|\begin{array}{ccccc|c} -c_1 & & & & e_1 & e_1 \\ e_2 & -c_2 & & & & \\ & \ddots & \ddots & & & \\ & & \ddots & -c_{m-1} & & \\ & & & e_m & -c_m & \\ \hline & & & & e & c \end{array}\right| = \left|\begin{array}{ccccc|c} -c_1 & & & & 0 & e_1 \\ e_2 & -c_2 & & & & \\ & \ddots & \ddots & & & \\ & & \ddots & -c_{m-1} & & \\ & & & e_m & -c_m & \\ \hline & & & & d & c \end{array}\right|$$

$$= c^1\prod_{i=1}^{m}(-c_i) + (-1)^{1+m+1}d^1\prod_{i=1}^{m}e_i.$$

If $|l| > 1$, then we have that

$$|P-\,tQ^T| = \left|\begin{array}{ccccc|ccccc} -c_1 & & & & e_1 & & & & & e_1 \\ e_2 & -c_2 & & & & & & & & \\ & \ddots & \ddots & & & & & & & \\ & & \ddots & -c_{m-1} & & & & & & \\ & & & e_m & -c_m & & & & & \\ \hline & & & & e & c & & & & \\ & & & & e & d & c & & & \\ & & & & e & & \ddots & \ddots & & \\ & & & & e & & & \ddots & c & \\ & & & & e & & & & d & c \end{array}\right|$$

$$= \left|\begin{array}{ccccc|ccccc} -c_1 & & & & 0 & & & & & e_1 \\ e_2 & -c_2 & & & & & & & & \\ & \ddots & \ddots & & & & & & & \\ & & \ddots & -c_{m-1} & & & & & & \\ & & & e_m & -c_m & & & & & \\ \hline & & & & d & c & & & & \\ & & & & 0 & d & c & & & \\ & & & & 0 & & \ddots & \ddots & & \\ & & & & 0 & & & \ddots & c & \\ & & & & 0 & & & & d & c \end{array}\right| = c^{|l|}\prod_{i=1}^{m}(-c_i) + (-1)^{|l|+m+1}d^{|l|}\prod_{i=1}^{m}e_i.$$

Next we calculate $|Q - tP^T|$ noticing that $e = c + d$. If $l = 0$, then we have that

$$|Q - tP^T| = \begin{vmatrix} -d_1 & e_2 & & & \\ & -d_2 & \ddots & & \\ & & \ddots & \ddots & \\ & & & -d_{m-1} & e_m \\ e_1 & & & & -d_m \end{vmatrix} = d^0 \prod_{i=1}^{m} (-d_i) + (-1)^{0+m+1} c^0 \prod_{i=1}^{m} e_i.$$

If $|l| = 1$, then we have that

$$|Q - tP^T| = \left|\begin{array}{ccccc|c} -d_1 & e_2 & & & & \\ & -d_2 & \ddots & & & \\ & & \ddots & \ddots & & \\ & & & -d_{m-1} & e_m & \\ e_1 & & & & -d_m & e \\ \hline e_1 & & & & & d \end{array}\right| = \left|\begin{array}{ccccc|c} -d_1 & e_2 & & & & \\ & -d_2 & \ddots & & & \\ & & \ddots & \ddots & & \\ & & & -d_{m-1} & e_m & \\ 0 & & & & -d_m & c \\ \hline e_1 & & & & & d \end{array}\right|$$

$$= d^1 \prod_{i=1}^{m} (-d_i) + (-1)^{1+m+1} c^1 \prod_{i=1}^{m} e_i.$$

If $|l| > 1$, then we have that

$$|Q - tP^T| = \left|\begin{array}{ccccc|ccccc} -d_1 & e_2 & & & & & & & & \\ & -d_2 & \ddots & & & & & & & \\ & & \ddots & \ddots & & & & & & \\ & & & -d_{m-1} & e_m & & & & & \\ e_1 & & & & -d_m & e & e & \cdots & e & e \\ \hline & & & & & d & c & & & \\ & & & & & & d & \ddots & & \\ & & & & & & & \ddots & \ddots & \\ & & & & & & & & d & c \\ e_1 & & & & & & & & & d \end{array}\right|$$

$$= \left|\begin{array}{ccccc|ccccc} -d_1 & e_2 & & & & & & & & \\ & -d_2 & \ddots & & & & & & & \\ & & \ddots & \ddots & & & & & & \\ & & & -d_{m-1} & e_m & & & & & \\ 0 & & & & -d_m & c & 0 & \cdots & 0 & 0 \\ \hline & & & & & d & c & & & \\ & & & & & & d & \ddots & & \\ & & & & & & & \ddots & \ddots & \\ & & & & & & & & d & c \\ e_1 & & & & & & & & & d \end{array}\right| = d^{|l|} \prod_{i=1}^{m} (-d_i) + (-1)^{|l|+m+1} c^{|l|} \prod_{i=1}^{m} e_i. \qquad \square$$

Now we calculate the Alexander polynomial $\Delta_K(t)$ of K diving the case into two depending on the value of l; $l \geq 0$ or $l < 0$. Here note the following.

ε	a	b	c	d	e
1	1	0	1	$-t$	$1-t$
-1	0	-1	t	-1	$-(1-t)$

ε_i	a_i	b_i	c_i	d_i	e_i
1	1	0	1	$-t$	$1-t$
-1	0	-1	t	-1	$-(1-t)$

Case. $l \geq 0$: From the above table, we have the following;

$$|P - t\,Q^T| = c^{|l|}\prod_{i=1}^{m}(-c_i) + (-1)^{|l|+m+1}d^{|l|}\prod_{i=1}^{m}e_i$$

$$= 1^{l}\,(-1)^p(-t)^{m-p} + (-1)^{l+m+1}(-t)^{l}(-1)^{m-p}(1-t)^m$$

$$= (-1)^{1-p}\,\{t^{l}(1-t)^m - (-t)^{m-p}\}$$

$$|Q - t\,P^T| = d^{|l|}\prod_{i=1}^{m}(-d_i) + (-1)^{|l|+m+1}c^{|l|}\prod_{i=1}^{m}e_i$$

$$= (-t)^{l}\,t^{p}\,1^{m-p} + (-1)^{l+m+1}1^{l}\,(-1)^{m-p}(1-t)^m = (-1)^{l+1-p}\,\{(1-t)^m - t^{l}(-t)^p\}$$

Case. $l < 0$: From the above table, we have the following;

$$|P - t\,Q^T| = c^{|l|}\prod_{i=1}^{m}(-c_i) + (-1)^{|l|+m+1}d^{|l|}\prod_{i=1}^{m}e_i$$

$$= t^{-l}\,(-1)^p(-t)^{m-p} + (-1)^{-l+m+1}(-1)^{-l}(-1)^{m-p}(1-t)^m$$

$$= (-1)^{1-p}\,\{(1-t)^m - t^{-l}(-t)^{m-p}\}$$

$$|Q - t\,P^T| = d^{|l|}\prod_{i=1}^{m}(-d_i) + (-1)^{|l|+m+1}c^{|l|}\prod_{i=1}^{m}e_i$$

$$= (-1)^{-l}\,t^{p}\,1^{m-p} + (-1)^{-l+m+1}t^{-l}\,(-1)^{m-p}(1-t)^m = (-1)^{-l+1-p}\,\{t^{-l}(1-t)^m - (-t)^p\}$$

In both cases, we obtain that $\Delta_K(t) \doteq \Delta_k(t)\,\{(1-t)^m - t^l(-t)^p\}\,\{(1-t)^m - t^{-l}(-t)^{m-p}\}$, and thus we complete the proof. □

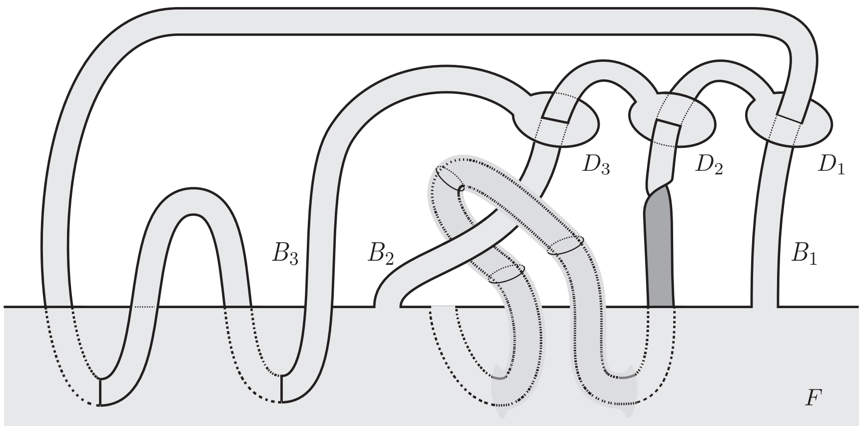

Fig. 9

Proof of Theorem 1.5. For each i $(1 \leq i \leq N)$, we can construct a simple-ribbon knot k_i with $\Delta_{k_i}(t) = \varphi(t\,; m_i, p_i, l_i)\,\varphi(t^{-1}\,; m_i, p_i, l_i)$ by following the proof of Theorem 1.1 (see also Figure 9). Let K^* be the connected sum of $k_1, k_2, \ldots, k_N$. Then K^* is a simple-ribbon knot

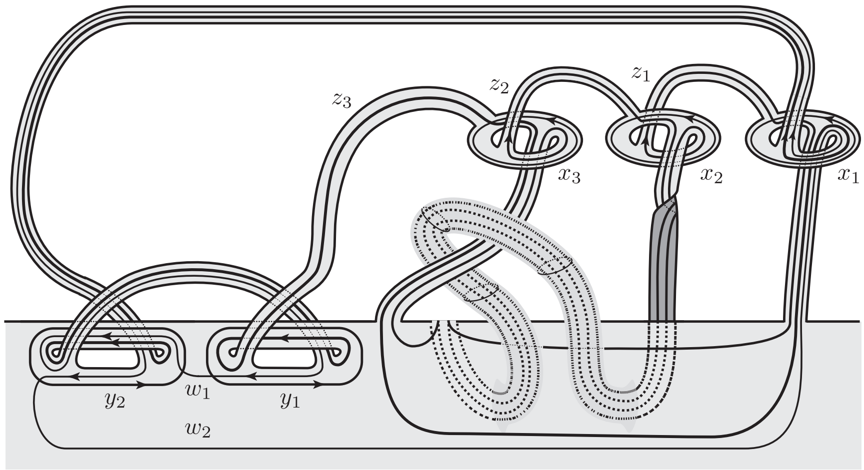

Fig. 10

such that $\Delta_{K^*}(t) = \Delta(t)$. Let $\mathcal{D} \cup \mathcal{B}$ be the set of disks and bands which gives the SR-fusion on the trivial knot $\mathcal{O} = \partial D_0$ producing K^*. Take a 3-ball H which is a small neighborhood of a point of $\mathcal{O} - \mathcal{B}$ and a trivial knot ρ in H which intersects D_0 twice so that $\mathrm{lk}(\rho, \mathcal{O}) = 2$. Let V^* be the closure of $S^3 - N(\rho; S^3)$. Since ρ is the trivial knot, V^* is an unknotted torus which contains K^* with $w_{V^*}(K^*) = 2$, where $w_{V^*}(K^*)$ is the absolute value of the algebraic intersection number of K^* with a meridian disk of V^*.

Let V be a tubular neighborhood of the Kinoshita-Terasaka knot κ and f a faithful homeomorphism of V^* onto V, i.e. f maps the preferred longitude of ∂V^* onto the preferred longitude of ∂V. Since $\Delta_\kappa(t) = 1$, we obtain that $\Delta_K(t) = \Delta_{K^*}(t)\,\Delta_\kappa(t^2) = \Delta_{K^*}(t) = \Delta(t)$ for $K = f(K^*)$ by Proposition 8.23 of [1]. Since f is faithful and both of K^* and κ are ribbon knots, K is also a ribbon knot by Lemma 3 of [8][1]. On the other hand, as $w_V(K) = w_{V^*}(K^*) = 2$ and κ is a non-trivial knot, K is not a simple ribbon knot by Corollary 1.8 of [5]. □

3. Proof of Theorem 1.6

Note that if K is a knot of $\mathcal{K}_m$, then $\det(K) = |\Delta_K(-1)| = (2^m - 1)^a (2^m + 1)^b$ for some non-negative integers a and b by Corollary 1.2. Moreover if K is also a knot of $\mathcal{K}_n$, then $\det(K) = (2^n - 1)^c (2^n + 1)^d$ for some non-negative integers c and d, and thus the set of prime factors of $(2^m - 1)^{a'} (2^m + 1)^{b'}$ and $(2^n - 1)^{c'} (2^n + 1)^{d'}$ coinside, where $i' = \min(i, 1)$ for a, b, c, and d.

Let $P(x)$ be the set of prime factors of an integer $x > 1$, and (y, z) the greatest common divisor of positive integers y and z. Note that if $P(y) = P(z)$ and $(y, z) = w$, then we have that $P(y) = P(z) = P(w)$. Here we prepare several lemmas, the first one of which is the theorem by P. Mihăilescu (the Catalan conjecture).

[1]Lemma 3 shows that K is ribbon cobordant to K^* if κ is a ribbon knot, although it states that K is cobordant to K^*.

Lemma 3.1 ([6, Theorem 5])**.** *The following equation has no other integer solutions but* $3^2 - 2^3 = 1$.

$$x^u - y^v = 1 \ (x > 0, y > 0, u > 1, v > 1) \tag{3.1}$$

Lemma 3.2 ([2, Theorem 1])**.** *Let* A, m, *and* n *be integers such that* $A > 1$ *and* $m > n \geq 1$. *Then* $P(A^m - 1) = P(A^n - 1)$ *if and only if* $m = 2$, $n = 1$, *and* $A = 2^l - 1$ *for an integer* $l > 0$.

Lemma 3.3. *Let* A *be an integer such that* $A > 1$. *Then the followings hold.*

(1) $P(A^p + 1) = P(A + 1)$ *for an odd integer* p (> 1) *if and only if* $p = 3$ *and* $A = 2$.
(2) $P(A^q - 1) = P(A + 1)$ *for an even integer* q (> 0) *if and only if* $q = 2$ *and* $A = 2^l + 1$ *for an integer* $l \geq 0$.

Proof. Since the if parts are easy to be checked, we only show the only if parts.
(1) First the following equation holds, since p is odd.

$$B = \frac{A^p + 1}{A + 1} = A^{p-1} - A^{p-2} + \cdots - A + 1 = \sum_{i=0}^{p-2} \binom{p}{i} (A + 1)^{p-i-1} (-1)^i + p \tag{3.2}$$

If p is prime, then we have that $(B, A + 1) = (A + 1, p) = p$ from equation (3.2), and thus that $P(B) = \{p\}$, since $P(B) \subset P(A^p + 1) = P(A + 1)$. Moreover, we have that $B \equiv p \pmod{p^2}$ also from equation (3.2), since $A + 1 \equiv 0 \pmod{p}$, $\binom{p}{p-2} \equiv 0 \pmod{p}$. Hence we obtain that $B = p$. If $p > 3$, then we also have that

$$B = A^{p-1} - A^{p-2} + \cdots - A + 1 = A(A-1)(A^{p-3} + A^{p-5} + \cdots + 1) + 1 > A(A-1)\frac{p-1}{2} + 1 \geq p, \tag{3.3}$$

since $A \geq 2$. However then it contradicts that $B = p$. Therefore we have that $p = 3$. Then we have that $A^2 - A + 1 = B = p = 3$ from equation (3.2), and thus that $A = 2$, since $A > 1$, which completes the proof.

If p is not prime, then let p' be a prime factor of p, and let $p = p'r$ and $B = A^r$. Since r and p' are odd, we have that $A + 1$ divides $A^r + 1 = B + 1$ and that $B + 1$ divides $B^{p'} + 1$. Hence we have that $P(A + 1) \subset P(B + 1) \subset P(B^{p'} + 1) = P(A^p + 1)$, since $B^{p'} = A^p$. Hence we have that $P(B^{p'} + 1) = P(B + 1)$, since $P(A^p + 1) = P(A + 1)$. Thus from the previous case, we have that $p' = 3$ and $B = A^r = 2$, and thus $A = 2$ and $r = 1$. However then, we have that $p = p'r = 3$, which contradicts that p is not prime.

(2) Since q is even, we have that $q \geq 2$. Hence we have that $P(A-1) \subset P(A^q - 1) = P(A+1)$, and thus that $P(A^2 - 1) = P((A-1)(A+1)) = P(A+1) = P(A^q - 1)$. Thus we have that $q = 2$ from Lemma 3.2. If $A \neq 2 = 2^0 + 1$, then we have that $A - 1 > 1$ and thus that $A + 1$ and $A - 1$ are not coprime, since $P(A-1) \subset P(A+1)$. Hence we have that $(A+1, A-1) = (A-1, 2) = 2$, since $A + 1 = (A - 1) + 2$. Therefore we obtain that $A - 1 = 2^l$ for $l > 0$, which completes the proof. □

Using Lemma 3.1 and Lemma 3.3, we show the following.

Proposition 3.4. *Let* A, m, *and* n *be integers such that* $A > 1$ *and* $m, n \geq 1$. *Then we have the following.*

(1) $P(A^m + 1) = P(A^n + 1)$ $(m > n)$ *if and only if* $m = 3$, $n = 1$, *and* $A = 2$;

(2) $P(A^m+1) = P(A^n-1)$ *if and only if one of the following holds.*

(i) $m = 1$, $n = 1$, *and* $A = 3$;

(ii) $m = 3$, $n = 2$, *and* $A = 2$;

(iii) $m = 2$, $n = 4$, *and* $A = 3$; *and*

(iv) $m = 1$, $n = 2$, *and* $A = 2^l + 1$ *for an integer* $l \geq 0$.

Proof. First we have the following for indeterminate X and positive integers s, t, and q and a non-negative integer r such that $s = qt + r$.

$$X^s \pm 1 = (X^t+1)(X^{s-t} - X^{s-2t} + \cdots - (-1)^q X^{s-qt}) + (-1)^q X^r \pm 1 \tag{3.4}$$

$$X^s + 1 = (X^t-1)(X^{s-t} + X^{s-2t} + \cdots + X^{s-qt}) + X^r + 1 \tag{3.5}$$

Let $g = (m, n)$. Then we have the following.

Claim 3.5. $(A^m+1, A^n+1), (A^m+1, A^n-1) = 1, 2$ or A^g+1.

Proof. For positive integers c_0 and c_1, let $(c_0, c_1) = (c_1, c_2) = \cdots = (c_{N-1}, c_N) = c_N$ be the sequence obtained by the Euclidian algorithm. Then letting $c_i = q_{i+1}c_{i+1} + q_{i+2}$, we also have the following from equations (3.4) and (3.5).

$$A^{c_{N-1}} \pm 1 = (A^{c_N}+1)(A^{c_{N-1}-c_N} - A^{c_{N-1}-2c_N} + \cdots - (-1)^q A^{c_{N-1}-q_Nc_N}) + (-1)^{q_N} A^0 \pm 1 \tag{3.6}$$

$$A^{c_{N-1}} + 1 = (A^{c_N}-1)(A^{c_{N-1}-c_N} + A^{c_{N-1}-2c_N} + \cdots + A^{c_{N-1}-q_Nc_N}) + A^0 + 1 \tag{3.7}$$

Hence by letting $(c_0, c_1) = (m, n)$ or (n, m), we have that (A^m+1, A^n+1), (A^m+1, A^n-1) is either A^g+1 or $(A^g \pm 1, 2)$, which induces the conclusion. □

Since the if parts are easy to be checked, we only show the only if parts.

(1) Since $P(A^m+1) = P(A^n+1)$, we have that A^m+1 and A^n+1 are not coprime, and thus that $(A^m+1, A^n+1) = 2$ or A^g+1 from Claim 3.5. In the former case, we have that $P(A^m+1) = P(A^n+1) = P(2) = \{2\}$. Thus, $A^m+1 = 2^k$ and $A^n+1 = 2$ for $k > 1$, since $m > n$. However then, we have that $A = 1$, which contradicts that $A > 1$. In the latter case, we have that $P(A^m+1) = P(A^n+1) = P(A^g+1)$ and that $m = gM$ with an odd integer M from equation (3.4). If $M = 1$, then $m = g$, which contradicts that $m > n$. Thus M is odd and $M > 1$. Then we have that $M = 3$ and $A^g = 2$ by Lemma 3.3 (1), and thus that $A = 2$, $g = 1$, $m = gM = 3$. Hence we have that $n = g = 1$, since $m > n$, which completes the proof.

(2) Since $P(A^m+1) = P(A^n-1)$, we have that A^m+1 and A^n-1 are not coprime, and thus that $(A^m+1, A^n-1) = 2$ or A^g+1 from Claim 3.5. In the former case, we have that $P(A^m+1) = P(A^n-1) = P(2) = \{2\}$, and thus that $A^m+1 = 2$ or $A^n-1 = 2$. If $A^m+1 = 2$, then $A^m = 1$, which contradicts that $A > 1$. If $A^m+1 = 2^k$ $(k > 1)$ and $A^n-1 = 2$, then we have that $A = 3$ and $n = 1$, and thus that $A^m+1 = 3^m+1 = 2^k$ $(k > 1)$. Then by Lemma 3.1, we have that $m = 1$, and thus obtain condition (i).

In the latter case, we have that $P(A^m+1) = P(A^n-1) = P(A^g+1)$ and that $m = gM$ with an odd integer M from equation (3.4). Consider the case where $M > 1$. Then we have that $M = 3$ and $A^g = 2$ by Lemma 3.3 (1), and thus that $A = 2$, $g = 1$, $m = gM = 3$. Since $A^m+1 = 2^3+1 = 9$, and thus $P(2^n-1) = P(9) = \{3\}$ and $(A^m+1, A^n-1) = (9, 2^n-1) = 3$,

Table 1. Ribbon knots with up to 10 crossings, where $F(t\,; m, p, l) = \varphi(t\,; m, p, l)\,\varphi(t^{-1}\,; m, p, l)$

K	simple-ribbon	$\delta_2(K)$	det(K)	$\Delta'_K(t)$
6_1	○	0	9	$F(t;2,0,0) = 2–5t + 2t^2$
8_8	○	5	25	$F(t;2,1,-1) = 2–6t + 9t^2–6t^3 + 2t^4$
8_9	○	7	25	$F(t;2,2,1) = 1–3t + 5t^2–7t^3 + 5t^4–3t^5 + t^6$
8_{20}	○	9	9	$F(t;2,1,0) = 1–2t + 3t^2–2t^3 + t^4$
9_{27}	○	5	49	$F(t;3,1,0) = 1–5t + 11t^2–15t^3 + 11t^4–5t^5 + t^6$
9_{41}	○	7	49	$F(t;3,0,0) = 3–12t + 19t^2–12t^3 + 3t^4$
9_{46}	○	0	9	$F(t;2,0,0) = 2–5t + 2t^2$
10_3	×	1	25	$6–13t + 6t^2$
10_{22}	×	11	49	$2–6t + 10t^2–13t^3 + 10t^4–6t^5 + 2t^6$
10_{35}	×	1	49	$2–12t + 21t^2–12t^3 + 2t^4$
10_{42}	○	9	81	$F(t;3,2,0) = 1–7t + 19t^2–27t^3 + 19t^4–7t^5 + t^6$
10_{48}	×	91	49	$1–3t + 6t^2–9t^3 + 11t^4–9t^5 + 6t^6–3t^7 + t^8$
10_{75}	○	9	81	$F(t;3,3,0) = 1–7t + 19t^2–27t^3 + 19t^4–7t^5 + t^6$
10_{87}	○	0	81	$F(t;3,2,-1) = 2–9t + 18t^2–23t^3 + 18t^4–9t^5 + 2t^6$
10_{99}	○	81	81	$F(t;1,1,1)F(t;2,2,0) = 1–4t + 10t^2–16t^3 + 19t^4–16t^5 + 10t^6–4t^7 + t^8$
10_{123}	×	1	121	$1–6t + 15t^2–24t^3 + 29t^4–24t^5 + 15t^6–6t^7 + t^8$
10_{129}	○	5	25	$F(t;2,1,1) = 2–6t + 9t^2–6t^3 + 2t^4$
10_{137}	○	1	25	$F(t;2,0,1) = 1–6t + 11t^2–6t^3 + t^4$
10_{140}	○	9	9	$F(t;2,1,0) = 1–2t + 3t^2–2t^3 + t^4$
10_{153}	○	35	1	$F(t;1,1,2) = 1–t–t^2 + 3t^3–t^4–t^5 + t^6$
10_{155}	○	7	25	$F(t;2,2,1) = 1–3t + 5t^2–7t^3 + 5t^4–3t^5 + t^6$
$3_1 \sharp 3_1^*$	○	9	9	$F(t;1,1,1) = 1–2t + 3t^2–2t^3 + t^4$
$4_1 \sharp 4_1$	○	1	25	$F(t;2,0,1) = 1–6t + 11t^2–6t^3 + t^4$
$5_1 \sharp 5_1^*$	×	121	25	$1–2t + 3t^2–4t^3 + 5t^4–4t^5 + 3t^6–2t^7 + t^8$
$5_2 \sharp 5_2^*$	×	1	49	$4–12t + 17t^2–12t^3 + 4t^4$

we have that $2^n - 1 = 3$ and thus that $n = 2$. Therefore we obtain condition (ii).

Next consider the case where $M = 1$, i.e., $m = g$. Hence let $n = mq$ $(q \geq 1)$ and $D = A^m$. Thus we have that $P(D+1) = P(D^q - 1)$ and that $(D+1, D^q - 1) = D + 1$. Therefore q is even, since otherwise $D+1$ does not divide $D^q - 1$. Then we have that $q = 2$ and $D = 2^l + 1$ for $l \geq 0$ by Lemma 3.3 (2). If $m > 1$ and $l > 1$, then the equation $A^m = 2^l + 1$ has the unique solution $(A, m, l) = (3, 2, 3)$ by Lemma 3.1, and thus we obtain condition (iii). If $m = 1$, then we have that $n = mq = 2$ and $A = 2^l + 1$ for $l \geq 0$, i.e., condition condition (iv). If $l = 0$ (resp. 1), then we have that $A = 2$ (resp. $A = 3$) and $m = 1$, and thus that condition (iv). □

Now using Proposition 3.4 and Lemma 3.2 we obtain the following.

Lemma 3.6. *Let p, q, r, s, M, N be positive integers with $M \neq N$. Then we have the following.*

(1) $(2^M - 1)^p \neq (2^N - 1)^r$.

(2) *If* $(2^M + 1)^q = (2^N + 1)^s$ $(M > N)$, *then* $M = 3$, $N = 1$, *and* $s = 2q$.

(3) *If* $(2^M + 1)^q = (2^N - 1)^r$, *then* $M = 3$, $N = 2$, $r = 2q$ *or* $M = 1$, $N = 2$, $q = r$.

(4) $(2^M - 1)^p(2^M + 1)^q \neq (2^N - 1)^r(2^N + 1)^s$

(5) *If* $(2^M - 1)^p(2^M + 1)^q = (2^N - 1)^r$, *then* $2M = N$, $p = q = r$.

(6) *If* $(2^M - 1)^p(2^M + 1)^q = (2^N + 1)^r$, *then* $M = 1$, $N = 3$, $q = 2r$.

Proof. Note that if positive integers X, Y and non-negative integers p, q satisfies the equation $X^p = Y^q$, then $P(X) = P(Y)$. The first three statements are obtained by Lemma 3.2, Proposition 3.4 (1), and Proposition 3.4 (2), respectively. For the last three statements, note that $P((2^M - 1)^p(2^M + 1)^q) = P(2^{2M} - 1)$. Therefore (4) and (5) are obtained by Lemma 3.2, and (6) is obtained by Proposition 3.4 (2). □

Proof of Theorem 1.6. Let K be a knot of $\mathcal{K}_m \cap \mathcal{K}_n$. Then we have that $\det(K) = (2^m - 1)^a(2^m + 1)^b = (2^n - 1)^c(2^n + 1)^d$ for some non-negative integers a, b, c, and d by Corollary 1.2. Thus we obtain the conclusion by Lemma 3.6. □

References

[1] G. Burde and H. Zieschang: Knots, Walter de Gruyter & Co., Berlin, 1985.

[2] T. Ishikawa, N. Ishida and Y. Yukimoto: *On prime factors of $A^n - 1$*, Amer. Math. Monthly **111** (2004), 243–245.

[3] A. Kawauchi: A survey of knot theory, Birkhäuser Verlag, Basel, 1996.

[4] K. Kishimoto, T. Shibuya and T. Tsukamoto: *Simple-ribbon fusions and genera of links*, J. Math. Soc. Japan **68** (2016), 1033–1045.

[5] K. Kishimoto, T. Shibuya and T. Tsukamoto: *Simple-ribbon concordance of knots*, to appear in Kobe J. Math.

[6] P. Mihăilescu: *Primary Cyclotomic Units and a Proof of Catalan's Conjecture*, J. Reine Angew. Math. **572** (2004), 167–195.

[7] H. Schubert: *Knoten und vollringe*, Acta Math. **90** (1953), 131–286.

[8] T. Shibuya: *On the cobordism of compound knots*, Math. Sem. Notes, Kobe Univ. **8** (1980), 331–337.

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